Two solutions for inviscid rotational flow with corner eddies

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The steady rotational flow of an inviscid fluid in a two-dimensional channel or a circular tube toward a sink is treated. The velocity distribution at infinity is approximated by a cosine curve (which is nearly parabolic) for the two-dimensional case, and is taken as exactly parabolic for the axisymmetric case. The dependence of vorticity on stream-function is assumed to be everywhere the same as it is for streamlines coming from infinity upstream. The resulting linear equations of motion are solved exactly. The solutions show the rather unusual features of separating streamlines and regions of closed flow (corner eddies).

It is well known that, for a viscous fluid flowing in a channel or pipe with an abrupt contraction, eddies occur at the corners immediately preceding that contraction. The occurrence of such eddies, and their shape and size, are probably influenced strongly by the rotationality of the flow far upstream, and the intention in this note is to show two simple solutions which throw a little light on this influence. Since we are interested here in the effect of upstream vorticity only, the direct effect of viscosity will be ignored, although the distribution of vorticity far upstream will be assumed to be the same (or nearly so) as it would be for a fluid with viscosity. The particular flow to be studied is that of an inviscid homogeneous fluid in a long channel or pipe toward a sink in the middle of a wall across the channel. Under these assumptions, exact solutions of the governing differential systems (assumed to apply to the entire region of flow) will be seen to show the existence of separating streamlines and regions of closed flow or corner eddies. (For another instance of corner eddies, in stratified flow, see Yih (1958).) No singular surfaces occur in the flow (nor could they, in view of the assumption made about the dependence of vorticity on stream-function), and the solutions to be given are unlikely to represent the corresponding flow of a real fluid at large Reynolds number; this is part of the penalty for neglecting viscous forces entirely.

Two-dimensional flow into a line sink

In this section we consider steady two-dimensional flow in a long channel with half-width equal to unity terminating in a wall with a symmetrically placed line sink. The origin of a system of Cartesian co-ordinates is taken at the sink, with the centreline of the channel as the x-axis (see figure 1).

If ψ is Lagrange's stream function, the equation governing steady two-dimensional flow of an inviscid fluid is

$$\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = f(\psi), \tag{1}$$

where $-f(\psi)$ represents the vorticity and depends on ψ alone. The velocity distribution far upstream from the sink should be parabolic if it is to be made the same as that for the laminar flow of a viscous fluid in a long channel. However, a parabolic distribution of the upstream velocity would make equation (1) non-linear and preclude the possibility of a simple solution. Since we are looking here

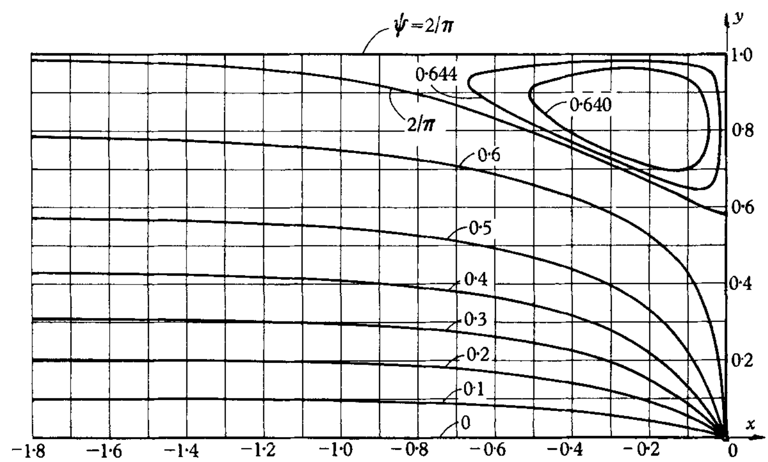

FIGURE 1. Pattern of two-dimensional flow into a sink.

for the effect of non-uniformity of the upstream velocity, a cosine distribution, which is nearly parabolic and makes equation (1) linear, can usefully be assumed. If the upstream velocity U is measured in terms of the centreline velocity $U_{\max}$, the dimensionless upstream velocity is then

$$u = U/U_{\max} = \cos\tfrac{1}{2}\pi y, \tag{2}$$

which vanishes at the walls ($y = \pm 1$). The corresponding (dimensionless) stream function far upstream is then

$$\psi = \int_0^y \cos\tfrac{1}{2}\pi y\,dy = (2/\pi)\sin\tfrac{1}{2}\pi y, \tag{3}$$

and the vorticity is $$-\nabla^2\psi = \tfrac{1}{2}\pi\sin\tfrac{1}{2}\pi y = \tfrac{1}{4}\pi^2\psi. \tag{4}$$

Thus the function $f(\psi)$ is equal to $-\frac{1}{4}\pi^2\psi$ not only far upstream but everywhere, and equation (1) becomes

$$\nabla^2\psi = -\tfrac{1}{4}\pi^2\psi. \tag{5}$$

The boundary conditions on ψ are

(i) $\psi \to (2/\pi)\sin\frac{1}{2}\pi y$ as $x \to -\infty$,

(ii) $\psi = \pm(2/\pi)$ for $y = \pm 1$,

(iii) $\psi = \mp(2/\pi)$ for $y \lessgtr 0$ and $x = 0$.

Solution of equation (5) by the method of separation of variables yields

$$\psi = (2/\pi)\sin\tfrac{1}{2}\pi y + \sum_{n=1}^{\infty} C_n \sin n\pi y \exp\{(n^2-\tfrac{1}{4})^{\frac{1}{2}}\pi x\}, \tag{6}$$

which satisfies (i) and (ii), and is an odd function of y. It is therefore only necessary to determine the C_n to satisfy

$$\sum_{n=1}^{\infty} C_n \sin n\pi y = (2/\pi)(1-\sin\tfrac{1}{2}\pi y) \quad (0 < y \leqslant 1), \tag{7}$$

which yields

$$C_n = \frac{4}{n\pi^2}\left(1+\frac{\cos n\pi}{4n^2-1}\right). \tag{8}$$

Equations (6) and (8) constitute the solution. The flow pattern for half of the channel is shown in figure 1, in which the corner eddy is evident. The point of separation is at infinity upstream.

It should be noted that, since the flow in the eddies does not originate at infinity, there is no *a priori* reason why equations (4) and (5) should govern the flow in the eddies. The details of the eddy flow may be significantly different from those given by equation (6) (see Batchelor 1956), and the solution (6) is likely to be valid for a viscous fluid at high Reynolds number only for the region outside the corner eddies.

Axisymmetric flow into a point sink

In this section we consider steady axisymmetric flow in a long circular pipe with unit radius terminating in a wall with a point sink at the centre. The point sink will be taken to be the origin of a set of cylindrical co-ordinates, with the centreline of the pipe as the z-axis and with the radial distance therefrom denoted by r (see figure 2).

If ψ is now Stokes's stream function, steady axisymmetric flow of an inviscid fluid is governed by the equation

$$\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2} = r^2F(\psi), \tag{9}$$

where $rF(\psi)$ represents the vorticity and F depends on ψ alone (Lamb 1945). The velocity distribution far upstream can here be assumed to be parabolic, as for the laminar flow of a viscous fluid in a long pipe, so that the dimensionless velocity far upstream is

$$w = W/W_{\max} = r^2-1. \tag{10}$$

The corresponding dimensionless stream function far upstream is

$$\psi = \tfrac{1}{2}r^2-\tfrac{1}{4}r^4, \tag{11}$$

and the vorticity there is $-2r$. Thus the function F is constant (-2) far upstream and hence everywhere, and the equation governing the flow everywhere is

$$\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2} = -2r^2. \tag{12}$$

The boundary conditions are

(i) $\psi \to \frac{1}{2}r^2 - \frac{1}{4}r^4$ as $z \to \infty$,
(ii) $\psi = \frac{1}{4}$ for $r = 1$,
(iii) $\psi = \frac{1}{4}$ for $z = 0$ $(0 < r \leqslant 1)$.

Solution of equation (12) by the separation of variables yields

$$\psi = \tfrac{1}{2}r^2 - \tfrac{1}{4}r^4 + \sum_{n=1}^{\infty} C_n r J_1(\lambda_n r) \exp(-\lambda_n z) \tag{13}$$

which satisfies (i) and (ii) if λ_n are the zeros of $J_1(\lambda)$. The coefficients C_n are determined by the condition (iii), and we find

$$C_n = \frac{1}{2\lambda_n J_0^2(\lambda_n)}\left(1 + \frac{8J_0(\lambda_n)}{\lambda_n^2}\right) \tag{14}$$

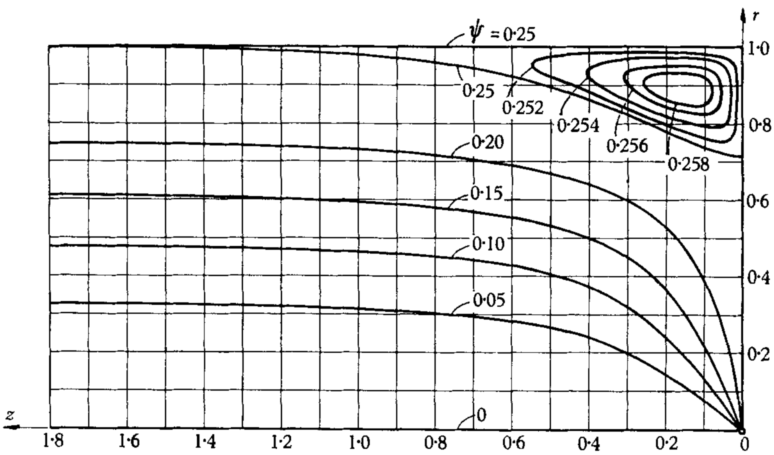

FIGURE 2. Pattern of axisymmetric flow into a sink.

by multiplying equation (13) (with $z = 0$) by $J_1(\lambda_m r)$, integrating between 0 and 1, and using the orthogonality of the functions $J_1(\lambda_n r)$. The detailed calculation for C_n involves integrations by parts and some known relationships for J_0 and J_1, and is omitted because it is straightforward. Equations (13) and (14) then constitute the solution.

The flow pattern for half of the meridianal plane is shown in figure 2, in which the ring-shaped corner eddy is evident. The point of separation is again at infinity. Again, the solution is strictly applicable to a fluid of small viscosity only outside the region of the ring eddy; equation (12) has been *assumed* to be valid for the whole field of flow, but the solution shows a region of closed flow where the vorticity is not determined by conditions far upstream and there is no *a priori* reason (other than convenience) why equation (12) should be applicable. We note that equation (12) does state that the vorticity is proportional to r everywhere, and therefore in the eddy in particular, which coincides with the exact result for flow with closed streamlines at high Reynolds number (Batchelor 1956); however, the constant of proportionality is also fixed by equation (12) and is unlikely to have here the value that would be required for a fluid with non-zero viscosity. As in the two-dimensional case, the flow shown in figure 2 is free from singular surfaces.

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